Int. J. Heat Mass Transfer, Vol. 25, No. 1, pp. 111–117, 1982.
Printed in Great Britain

0017–9310/82/010111–07 $03.00/0
Pergamon Press Ltd.

VARIATIONAL BOUNDS FOR THE EFFECTIVE THERMAL CONTACT RESISTANCE BETWEEN BODIES WITH ROUGH SURFACES

M. BOBETH and G. DIENER
Department of Physics, Technical University, 8027 Dresden, G.D.R.

(*Received* 20 *June* 1980 *and in revised form* 22 *June* 1981)

Abstract—The present paper is devoted to the estimation of the contact resistance between rough surfaces. The roughness is described by a stochastically varying local contact resistance. General variational principles are derived which permit to determine lower and upper bounds of the macroscopic contact resistance. To illustrate the method, it has been used to calculate an upper bound for a random arrangement of circular contact spots of equal size. The bound of the macroscopic contact resistance is obtained as a functional of the pair distribution function of the contact spots.

NOMENCLATURE

A, area;
a, area fraction of the contacts;
f, arbitrary function of position;
g, correlation function;
h, Fourier transformed function f;
J, integral operator defined by equation (3.14);
J^{-1}, inversion operator of J;
K, thermal conductivity;
K_+, thermal conductivity of the upper body;
K_-, thermal conductivity of the lower body;
$\mathbf{k}$, vector of the Fourier space;
L, height of a circular cylinder;
L_{st}, stochastic operator defined by equation (3.18);
N, number of contacts;
P, projection operator defined by equation (3.5);
Q, heat flow;
q, z-component of the heat flow density;
$\tilde{q}$, trial function of q;
q_0, constant heat flow density;
q^s, heat flow density of a single contact given by equation (4.2);
$\mathbf{R}$, 2-dim. position vector (x, y);
R_c, radius of a circular cylinder;
R_0, radius of a circular contact spot;
$\mathbf{r}$, 3-dim. position vector (x, y, z);
T, temperature field;
$\tilde{T}$, trial function of T;
ΔT, jump in temperature;
U, thermal conductance of a single circular contact;
U_{eff}, effective thermal conductance;
u_{st}, thermal conductance per unit area;
u_{eff}, effective value of u_{st};
V, volume;
∂V, surface of volume V;
w_{st}, reduced local contact resistance defined by equation (3.17);
w_{eff}, effective value of w_{st};
w_+, upper bound of w_{eff};
w_-, lower bound of w_{eff}.

Greek symbols

ε, function of the area fraction defined by equation (4.20);
θ, step function defined by equation (4.3);
v^s, abbreviation of expression Jq^s in equation (4.4);
ρ, dimensionless integration variable.

1. INTRODUCTION

THE PHENOMENON of heat transfer between two bodies in contact is of great interest in many technical fields and, therefore, considerable attention has been paid to this problem during the last decades. The thermal contact resistance can be caused by various factors (see e.g. [1]). In this paper we are dealing with the influence of the roughness and waviness on the heat transfer. In consequence of the roughness, the two bodies are in direct contact only at some spots (see Fig. 1) and, therefore, the heat flow is constricted near the surfaces. This effect is one of the main causes of thermal contact resistance. Previous estimations of the so-called constriction resistance have been based on special assumptions concerning the shape of single contact spots as well as their arrangement. For an isolated flat circular contact between two halfspaces the constriction resistance has been calculated exactly [2]. This result can

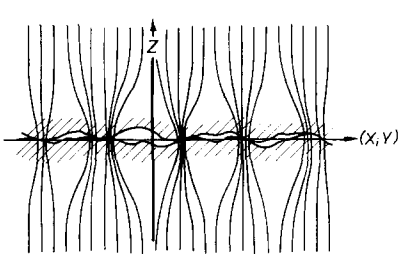

FIG. 1. Constriction of the heat flow near the rough surfaces.

be used to estimate the constriction resistance of two bodies with many circular contact spots by means of a parallel connection. This is a good approximation if the mutual distances of the contact spots are much larger than their characteristic size. In the opposite case, the mutual influence of the contacts must be taken into consideration. For a regular arrangement of contacts this can be done by considering the bodies as a parallel connection of identical finite cells, each of them containing only one contact spot. In order to simplify the calculations, these cells are usually approximated by two circular cylinders connected by a coaxial circular contact [1, 3, 4]. If, beside the roughness, there is also a waviness of the surfaces, the contacts are arranged in clusters. The constriction resistance of such clusters has been studied in [5–8] where circular contacts of different sizes are admitted.

The present work proposes another approach to the problem of constriction resistance. Starting from a stochastic 'mesoscopic' description of the contact resistance, we can formulate general variational principles, which enable us, in principle, to derive upper and lower bounds for the macroscopic constriction resistance. The actual value of the resistance has to lie between these bounds. To obtain satisfactory bounds which are narrow enough, we have to construct suitable trial functions for the mesoscopic heat-flow vector and the temperature, respectively, and to insert them into the corresponding variational principles. An analogous approach has been largely used in the theory of effective material parameters of composites and other random media [9, 10]. The advantage of this method consists in the fact, that it yields rigorous inequalities of general validity for the effective mascroscopic quantities.

After a general definition of the effective contact resistance, given in Section 2, the variational principles are derived in Section 3. In Section 4 the formulism is applied to obtain an upper bound for the contact resistance of a random arrangement of circular contact spots. Informations about the arrangement are involved in form of the pair distribution function.

Let us mention that all results obtained here are also valid for the electric contact resistance because of the mathematical equivalence of both problems.

2. LOCAL AND EFFECTIVE THERMAL CONDUCTANCE

We start from the well known definition of the macroscopic or effective thermal conductance U_{eff}:

$$Q = -U_{\text{eff}}\,\Delta T \tag{2.1}$$

where Q is the heat flow between the bodies in contact and ΔT is the jump in temperature at the interface. The total heat flow is given as a surface integral over the heat flow density q which is determined by

$$q(\mathbf{R}) = -u_{\text{st}}(\mathbf{R})\,\Delta T(\mathbf{R}) \tag{2.2}$$

$$Q = \int \mathrm{d}^2\mathbf{R}\, q(\mathbf{R}). \tag{2.3}$$

The vector $\mathbf{R}$ denotes the position at the interface. Because of the roughness of the surfaces the local thermal conductance per unit area $u_{\text{st}}(\mathbf{R})$ is a stochastic function of position as indicated by the subscript 'st'. Inside the areas of direct contact, u_{st} takes very high values whereas outside the contacts it is small. In consequence of the constriction effect caused by the variations of u_{st} the heat flow density q and the jump in temperature ΔT are also random functions of position. The macroscopic value of ΔT in equation (2.1) has to be understood as the mean value averaged over the interface.

In general, the rough surface differs only slightly from a plane, that means the deviations from the plane are small compared to the characteristic extensions or correlation lengths of the contact spots and, therefore, we approximate the two bodies in contact by two homogeneous halfspaces bounded by the plane $z = 0$. Thus the roughness and the partial contact are expressed only by means of the randomly varying local conductance u_{st}. Finally, the macroscopic or effective thermal conductance per unit area u_{eff} is defined, analogously to equations (2.1) and (2.2), by

$$u_{\text{eff}} = \frac{U_{\text{eff}}}{A} = -\frac{\int \mathrm{d}^2\mathbf{R}\, q(\mathbf{R})}{\int \mathrm{d}^2\mathbf{R}\,\Delta T(\mathbf{R})} = -\frac{\langle q(\mathbf{R})\rangle}{\langle \Delta T(\mathbf{R})\rangle} \tag{2.4}$$

where the angle brackets

$$\langle \ldots \rangle = \frac{1}{A}\int \mathrm{d}^2\mathbf{R} \ldots \tag{2.5}$$

denote the average over a very large area A of the x, y-plane.

3. DERIVATION OF VARIATIONAL PRINCIPLES

We start from the basic equation of a stationary heat flow

$$\operatorname{div} \mathbf{q}(\mathbf{r}) = 0. \tag{3.1}$$

Inside the bodies, the heat flow density is given by

$$\mathbf{q}(\mathbf{r}) = -K \operatorname{grad} T(\mathbf{r}). \tag{3.2}$$

If we assume the thermal conductivity K to be constant, equations (3.1) and (3.2) give

$$\operatorname{divgrad} T(\mathbf{r}) = 0. \tag{3.3}$$

The interface $z = 0$ has to be excluded since equation (3.2) does not hold there. Equation (3.3) determines together with the boundary condition (2.2) the temperature field $T(\mathbf{r})$.

Now we introduce a mean value $\langle T\rangle$ of the temperature, which is defined as an average over the plane $z = \text{const.}$ and, therefore, depends only upon the coordinate z. Thus the total temperature field may be written as

$$T(\mathbf{r}) = \langle T\rangle + PT \tag{3.4}$$

where the operator

$$Pf = f(\mathbf{r}) - \langle f\rangle \tag{3.5}$$

projects out the deviations of a stochastic function

from its mean value. From equation (3.3) it can easily be seen that the mean value $\langle T \rangle$ is a linear function of the z coordinate except the plane $z = 0$ where it suffers a jump. For the fluctuating part we obtain from equation (3.3)

$$\text{divgrad } PT(\mathbf{r}) = 0, \quad z \neq 0. \tag{3.6}$$

Because of the geometrical symmetry of the problem the field $q(\mathbf{r})$ is symmetric with respect to the interface $z = 0$. Moreover we can put

$$PT(\mathbf{R}, z) = -PT(\mathbf{R}, -z) \tag{3.7}$$

provided that the values of K are the same in both bodies. For two bodies of different thermal conductivities we have instead of equation (3.7)

$$K_+ PT(\mathbf{R}, z) = -K_- PT(\mathbf{R}, -z), \quad z > 0 \tag{3.8}$$

where K_+ and K_- denote the values of the thermal conductivity for $z > 0$ and $z < 0$, respectively. Thus in the following all calculations may be restricted to the upper halfspace $z > 0$. Using an integral theorem of potential theory, we transform equation (3.6) into

$$PT(\mathbf{r}) = \frac{1}{4\pi} \oiint_{\partial V} \mathrm{d}\mathbf{A}' \left\{ \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}-\mathbf{r}'|^3} PT(\mathbf{r}') + \frac{1}{|\mathbf{r}-\mathbf{r}'|} \frac{\partial}{\partial \mathbf{r}'} PT(\mathbf{r}') \right\}, \quad \mathbf{r} \in V. \tag{3.9}$$

This equation holds for every part V of the upper body and any $\mathbf{r}$ inside V. In particular, we can choose the whole upper body. In order to simplify the argumentation let us assume the body to be a large circular cylinder of radius R_c bounded by two planes $z = 0$ and $z = L$. Since the fluctuations PT tend to zero at large distances from the plane $z = 0$, only that parts of the surface of V which have sufficiently small values of z contribute to the integral. Therefore we can disregard the plane $z = L$. Moreover, if we require the lateral surface of the cylinder to be heat isolated, the second term of the integral vanishes there. The first term integrated over this part of the surface also vanishes for $R_c \to \infty$ because the integrand behaves as R_c^{-2} whereas the contributing surface increases only as R_c. Consequently, in equation (3.9) only the integral over the plane $z = +0$ remains

$$PT(\mathbf{r}) = \frac{1}{4\pi} \iint \mathrm{d}x'\mathrm{d}y' \left\{ \frac{z}{|\mathbf{r}-\mathbf{r}'|^3} PT(\mathbf{r}') - \frac{1}{|\mathbf{r}-\mathbf{r}'|} \frac{\partial}{\partial z'} PT(\mathbf{r}') \right\}_{z'=+0}. \tag{3.10}$$

If we introduce the notations

$$T(\mathbf{R}) = T(x, y, z = +0) = \frac{1}{2} \Delta T(\mathbf{R}) \tag{3.11}$$

$$q(\mathbf{R}) = -K \frac{\partial}{\partial z} T(x, y, z = +0) \tag{3.12}$$

and then go to the limit $z = +0$, then equation (3.10) becomes

$$PT(\mathbf{R}) = \frac{1}{2} PT(\mathbf{R}) + \frac{1}{4\pi} \int \mathrm{d}^2\mathbf{R}' \frac{1}{|\mathbf{R}-\mathbf{R}'|} \frac{1}{K} Pq(\mathbf{R}') \tag{3.13}$$

or

$$PT(\mathbf{R}) = \frac{1}{K} \int \mathrm{d}^2\mathbf{R}' \frac{1}{2\pi|\mathbf{R}-\mathbf{R}'|} Pq(\mathbf{R}') = :\frac{1}{K} JPq \tag{3.14}$$

where, for abbreviation, an integral operator J is defined by the last equality.

The heat flow density q which appears in equation (3.14) is related to the temperature jump at the interface by equation (2.2) which yields, together with equation (3.11)

$$T(\mathbf{R}) = -\frac{1}{2u_{st}(\mathbf{R})} q(\mathbf{R}). \tag{3.15}$$

Inserting this relation into the left-hand side of equation (3.14), we get

$$\left(\frac{K}{2u_{st}} + JP \right) q = \left\langle \frac{K}{2u_{st}} \right\rangle q \quad . \tag{3.16}$$

Let us define a reduced local contact resistance w_{st} and its effective value w_{eff} by

$$w_{st} = \frac{K}{2u_{st}}, \quad w_{eff} = \frac{K}{2u_{eff}} = \frac{KA}{2U_{eff}}. \tag{3.17}$$

Then equation (3.16) can be written as

$$L_{st} q := (w_{st} + JP) q = w_{eff} \langle q \rangle. \tag{3.18}$$

The last equality in (3.17) represents only a transcription of equation (2.4). Hence w_{eff} is proportional to the macroscopic contact resistance. In principle, equation (3.18) completely determines the heat flow density at the interface as well as the effective conductance u_{eff} (or w_{eff}) and, therefore, it may be considered as basic equation of our problem. Contrary to equations (3.2) and (3.3) it represents a 2-dim. formulation which, moreover, already contains the boundary condition (2.2).

The integral equation (3.18) can be transformed into a variational principle with the aid of a method worked out in [9, 10]. For this end we make use of the self-adjointness and the positive definiteness of the stochastic operator L_{st} defined in equation (3.18). For arbitrary functions $f_1(\mathbf{R})$ and $f_2(\mathbf{R})$ the relations

$$\langle f_1 L_{st} f_2 \rangle = \langle f_2 L_{st} f_1 \rangle, \tag{3.19}$$

$$\langle f_1 L_{st} f_1 \rangle \geq 0 \tag{3.20}$$

hold. The self-adjointness (3.19) can be seen immediately from the definitions (3.14) and (3.18). In order to prove the second property (3.20) we start with the examination of the expression

$$\langle fJPf \rangle = \langle (\langle f \rangle + Pf) JPf \rangle = \langle PfJPf \rangle. \tag{3.21}$$

By means of a Fourier transformation

$$h(\mathbf{k}) = \int \mathrm{d}^2\mathbf{R}\, \mathrm{e}^{-i\mathbf{k}\mathbf{R}} Pf(\mathbf{R}) \tag{3.22}$$

we can rewrite it as

$$\langle fJPf\rangle = \int \mathrm{d}^2\mathbf{k}\,\frac{1}{k}\,|h(\mathbf{k})|^2 \geq 0. \tag{3.23}$$

Obviously, expression (3.23) is positive. Thus we have shown JP to be a positive definite operator. This result leads us, together with $w_{st} \geq 0$, to equation (3.20).

Now let us replace function f_1 in (3.20) by $\tilde{q} - q$, where $\tilde{q}$ and q mean an arbitrary trial function and the exact solution of equation (3.18), respectively:

$$\langle(\tilde{q}-q)\,L_{st}(\tilde{q}-q)\rangle \geq 0, \tag{3.24}$$

$$\langle\tilde{q}L_{st}\tilde{q}\rangle \geq \langle\tilde{q}L_{st}q\rangle + \langle qL_{st}\tilde{q}\rangle - \langle qL_{st}q\rangle. \tag{3.25}$$

Using the relations (3.18) and (3.19) and requiring $\langle\tilde{q}\rangle = \langle q\rangle$ we can transform (3.25) in the following way

$$\begin{aligned}\langle\tilde{q}L_{st}\tilde{q}\rangle &\geq 2\langle\tilde{q}L_{st}q\rangle - \langle qL_{st}q\rangle \\ &= 2\langle\tilde{q}\rangle w_{eff}\langle q\rangle - \langle q\rangle w_{eff}\langle q\rangle \\ &= w_{eff}\langle\tilde{q}\rangle^2\end{aligned}$$

or

$$\frac{\langle\tilde{q}(w_{st}+JP)\tilde{q}\rangle}{\langle\tilde{q}\rangle^2} = :\ w_+ \geq w_{eff}. \tag{3.26}$$

For any $\tilde{q}$, the inequality provides us an upper bound of the effective local contact resistance. The bound w_+ coincides with the actual value w_{eff} only if $\tilde{q}$ is replaced by the exact solution q. Thus, equation (3.26) represents a variational principle which is completely equivalent to the integral equation (3.18). In order to obtain useful bounds, we have to construct trial functions $\tilde{q}$ which fit q sufficiently well.

A lower bound of the effective local contact resistance can also be derived from (3.14) and (3.15) if we eliminate the heat flow density instead of the temperature. In analogy to equation (3.18) we get

$$(w_{st}^{-1} + J^{-1}P)T = \langle w_{st}^{-1}\,T\rangle = w_{eff}^{-1}\,\langle T\rangle \tag{3.27}$$

where J^{-1} means the inversion of the integral operator J given by

$$J^{-1}f = -\int \mathrm{d}^2\mathbf{R}'\,\frac{1}{2\pi|\mathbf{R}-\mathbf{R}'|}\,\mathrm{divgrad}f(\mathbf{R}'). \tag{3.28}$$

Starting from this equation instead of equation (3.18), we can repeat the above considerations where L_{st} has to be substituted by the operator $w_{st}^{-1} + J^{-1}P$ which is positive definite and self-adjoint, too. The result is an upper bound of w_{eff}^{-1}

$$\frac{\langle\tilde{T}(w_{st}^{-1} + J^{-1}P)\tilde{T}\rangle}{\langle\tilde{T}\rangle^2} = :\ \frac{1}{w_-} \geq \frac{1}{w_{eff}} \tag{3.29}$$

for every trial function $\tilde{T}$. In other words, w_- represents a lower bound of w_{eff}. Thus, the two variational principles (3.26) and (3.29) enables us to confine the possible values of the effective contact resistance to an interval between an upper and a lower bound

$$w_- \leq w_{eff} \leq w_+ \tag{3.30}$$

where the bounds involve, of course, only partial information about the statistics of the local contact resistance.

4. AN UPPER BOUND FOR RANDOMLY DISTRIBUTED CIRCULAR CONTACTS

In this section we apply the variational principle to the special case of circular contacts of equal size distributed at random. Using the local contact resistance w_{st}, a perfectly conducting circular contact of radius R_0 can be described by

$$w_{st}(\mathbf{R}) = \begin{cases}0 & \text{for } |\mathbf{R}| \lessgtr R_0\\ \infty\end{cases} \tag{4.1}$$

where the origin of coordinates is placed at the centre of the circle. If we consider a single contact between two infinite halfspaces, the heat flow vector can be determined exactly [11]

$$q(\mathbf{R}) = q^s(\mathbf{R}) = q_0(1-R^2/R_0^2)^{-1/2}\,\theta(R_0-R) \tag{4.2}$$

where the constant q_0 is proportional to the total heat flow and θ is the step function

$$\theta(R) = \begin{cases}0 & \text{for } R \lessgtr 0.\\ 1\end{cases} \tag{4.3}$$

Application of the operator J to the heat flow vector q^s yields [11]

$$\begin{aligned}Jq^s &= q_0R_0\{\theta(R_0 - R)\,\pi/2 + \theta(R - R_0)\arcsin(R_0/R)\\ &= v^s.\end{aligned} \tag{4.4}$$

The constriction resistance of a single circular contact is given by [2]:

$$\frac{1}{U} = \frac{1}{2KR_0}. \tag{4.5}$$

If we have N such contacts between both halfspaces, the effective conductance may be roughly estimated taking simply their parallel connection

$$w_{eff} = \frac{KA}{2U_{eff}} \approx \frac{KA}{2UN} = \frac{A}{4NR_0} = \frac{\pi R_0}{4a} \tag{4.6}$$

where A denotes the nominal area of the interface and $a = N\pi R_0^2/A$ is the area fraction of the contacts.

An approximation which partly takes into account the mutual influence of the contacts can be derived from the variational principle (3.26). As trial function $\tilde{q}$ we choose the superposition of the fields of isolated contacts q^s

$$\tilde{q} = \sum_i q^s(\mathbf{R}-\mathbf{R}_i) \tag{4.7}$$

with

$$\langle\tilde{q}\rangle = 2q_0a. \tag{4.8}$$

$\mathbf{R}_i$ denotes the centre of the ith contact. Inserting the field (4.7) into equation (3.26), we find

$$w_+\langle\tilde{q}\rangle^2 = \langle\tilde{q}w_{st}\tilde{q}\rangle + \langle\tilde{q}J\tilde{q}\rangle - \langle\tilde{q}\rangle\,\langle J\tilde{q}\rangle. \tag{4.9}$$

Instead of equation (4.1) we now assume a large but finite value of the local contact resistance w_{st} outside the contacts. Since the trial function $\tilde{q}$ is zero there, the

first term of the right-hand side of equation (4.9) vanishes. Splitting up the summation in the second term, we write

$$w_+\langle\tilde{q}\rangle^2 = \sum_i \langle q^s(\mathbf{R}-\mathbf{R}_i)\, v^s(\mathbf{R}-\mathbf{R}_i)\rangle + \sum_{i\neq j} \langle q^s(\mathbf{R}-\mathbf{R}_i)\, v^s(\mathbf{R}-\mathbf{R}_j)\rangle - \langle\tilde{q}\rangle\langle J\tilde{q}\rangle^*. \quad (4.10)$$

The average $\langle q^s(\mathbf{R}-\mathbf{R}_i)\, v^s(\mathbf{R}-\mathbf{R}_j)\rangle$ depends only upon the difference vector $(\mathbf{R}_i-\mathbf{R}_j)$. In the case of a large number N of contacts we can replace the summation over the pairs $i \neq j$ by an integration over the difference vector $(\mathbf{R}_i-\mathbf{R}_j) \to \mathbf{R}$ weighted by a pair distribution function $p(\mathbf{R})$ which is normalized to $N-1$ where p means the probability density to find a contact at a distance $\mathbf{R}$ from a fixed one. At large distances the distribution of contacts becomes uncorrelated and, therefore, $p(\mathbf{R})$ tends to N/A. Hence we introduce a correlation function $g(\mathbf{R})$ in the usual way by

$$P(\mathbf{R}) = \frac{N}{A}[1+g(\mathbf{R})] \quad (4.11)$$

where g vanishes at infinity.

Using this function and taking into account the meaning of the angle brackets (2.5), we may write the double sum in equation (4.10) as

$$\begin{aligned}
&\sum_{i\neq j} \langle q^s(\mathbf{R}-\mathbf{R}_i)\, v^s(\mathbf{R}-\mathbf{R}_j)\rangle \\
&= \sum_{i\neq j} \frac{1}{A}\int \mathrm{d}^2\mathbf{R} q^s(\mathbf{R}-\mathbf{R}_i+\mathbf{R}_j)\, v^s(\mathbf{R}) \\
&= \frac{N}{A}\int \mathrm{d}^2\mathbf{R}' p(\mathbf{R}')\int \mathrm{d}^2\mathbf{R} q^s(\mathbf{R}-\mathbf{R}')\, v^s(\mathbf{R}) \\
&= \frac{N^2}{A^2}\int \mathrm{d}^2\mathbf{R}\mathrm{d}^2\mathbf{R}'(1+g(\mathbf{R}'))q^s(\mathbf{R}-\mathbf{R}')\, v^s(\mathbf{R}) \\
&= \frac{N^2}{A^2}\int \mathrm{d}^2\mathbf{R}\mathrm{d}^2\mathbf{R}' q^s(\mathbf{R}')\, v^s(\mathbf{R}) \\
&= \frac{N^2}{A^2}\int \mathrm{d}^2\mathbf{R}\mathrm{d}^2\mathbf{R}' g(\mathbf{R}')q^s(\mathbf{R}-\mathbf{R}')\, v^s(\mathbf{R}). \quad (4.12)
\end{aligned}$$

The last term in equation (4.10) gives

$$\langle\tilde{q}\rangle\langle J\tilde{q}\rangle = \frac{N^2}{A^2}\int \mathrm{d}^2\mathbf{R} q^s(\mathbf{R})\int \mathrm{d}^2\mathbf{R}' v^s(\mathbf{R}'). \quad (4.13)$$

With the aid of equations (4.12) and (4.13) we find for (4.10)

$$w_+\langle\tilde{q}\rangle^2 = \frac{N}{A}\int \mathrm{d}^2\mathbf{R} q^s(\mathbf{R})\, v^s(\mathbf{R}) + \frac{N^2}{A^2}\int \mathrm{d}^2\mathbf{R} g(\mathbf{R})\int \mathrm{d}^2\mathbf{R}' q^s(\mathbf{R}-\mathbf{R}')\, v^s(\mathbf{R}'). \quad (4.14)$$

The integral over $\mathbf{R}'$ can bed expressed in the form

$$\int \mathrm{d}^2\mathbf{R}' q^s(\mathbf{R}-\mathbf{R}')\, v^s(\mathbf{R}') = q_0^2 R_0^3 f\left(\frac{|\mathbf{R}|}{R_0}\right) \quad (4.15)$$

with

$$f(\rho) = \int \mathrm{d}^2\rho'(1-\rho'^2)^{-1/2}\,\theta(1-\rho'^2) \times \left\{\theta(1-(\rho-\rho')^2)\frac{\pi}{2} + \theta[(\rho-\rho')^2-1] \times \arcsin|\rho-\rho^1|^{-1}\right\} \quad (4.16)$$

$$f(0) = \pi^2. \quad (4.17)$$

This function has been evaluated numerically. The result is plotted in Fig. 2. The asymptotic behaviour is given by

$$f(\rho) = \frac{1}{2\pi}\left(\rho^{-1} + \frac{1}{3}\rho^{-3}\right) + O(\rho^{-5}). \quad (4.18)$$

From equations (4.8), (4.14) and (4.15) we finally obtain

$$w_+ = \frac{\pi}{4a} R_0 \varepsilon(a) \quad (4.19)$$

with

$$\varepsilon(a) = 1 + \frac{a}{\pi^3 R_0^2}\int \mathrm{d}^2\mathbf{R} g(\mathbf{R}) f\left(\frac{R}{R_0}\right). \quad (4.20)$$

If the correlation function is independent of the orientation, equation (4.20) simplifies to

$$\varepsilon(a) = 1 + \frac{2a}{\pi^2 R_0^2}\int \mathrm{d}R R\, g(R) f\left(\frac{R}{R_0}\right). \quad (4.21)$$

According to equation (4.20) the upper bound consists of two terms. The first one describes the parallel connection of contacts [equation (4.6)] and the second term represents a correction due to the interaction of the contacts.

The sign and value of the interaction term depend on the correlation function g. General features of $g(\mathbf{R})$ are the asymptotic vanishing and its behaviour at small values of the argument

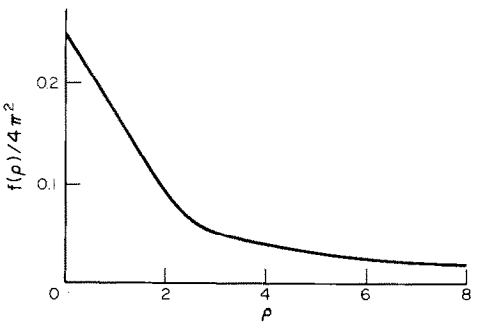

FIG. 2. Numerical calculation of the function f defined by equation (4.16).

* In order to prevent mathematical difficulties in performing the average of $J\tilde{q}$, we may add to (4.4) a convergence factor $\exp(-\varepsilon R)$, $\varepsilon \to +0$ which will be omitted in the final result.

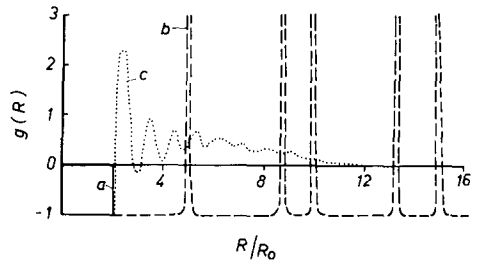

FIG. 3. Special correlation functions (schematically).

$$g(\mathbf{R}) = -1 \quad \text{for} \quad R < 2R_0 \tag{4.22}$$

which signifies that there is no overalpping of contact spots. In order to discuss the upper bound (4.19) qualitatively, we consider the following limiting cases which are shown schematically in Fig. 3.

(a) The correlation is negligible except condition (4.22), i.e. the function g is positive nowhere and the interaction term becomes negative. From equation (4.21) we get

$$\varepsilon(a) = 1 - 2.3a. \tag{4.23}$$

This assumption concerning g is reasonable only for small a.

(b) The contacts are arranged in a lattice. Thus extreme distances between neighbouring contacts are realised. The correlation function shows δ-like peaks and is equal to -1 elsewhere. Then the negative contributions of the integral are dominant and the bound will be relatively low. For a triangular lattice the function $\varepsilon(a)$ is plotted in Fig. 4.

(c) In the opposite case of clustering of contacts the correlation function g has positive peaks near $R = 2R_0$. This leads to a positive interaction term which rises with increasing cluster size.

Previous authors have already calculated the contact resistance of a regular arrangement of contact spots for small values of the area fraction a. For comparison the result of Holm [2] for a square lattice of circular contact spots is also plotted in Fig. 4. It lies everywhere below the upper bound obtained here.

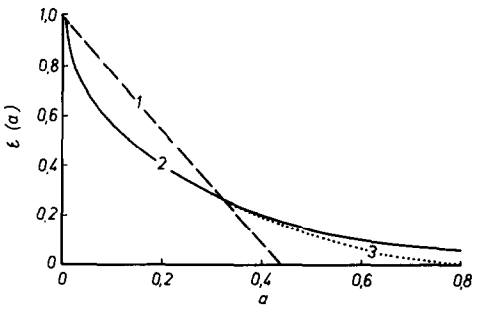

FIG. 4. Function $\varepsilon(a)$: (1) for the upper bound (4.19) of uncorrelated circular contact spots; (2) for the upper bound (4.19) of a triangular lattice of circular contact spots; (3) estimation by Holm [2] for a square lattice of circular contact spots.

However, for higher area fractions of the contacts this estimation is not applicable whereas the upper bound (4.19) remains valid there. Other estimations which lie above the upper bound can be excluded.

5. CONCLUSION

In this paper a general procedure for calculating lower and upper bounds of the effective electric or thermal contact resistance by means of variational principles has been presented. This method can be applied to a great number of practical cases. The only approximation which has been made to formulate the variational principles is the replacement of the rough surface by a plane. The random local contact resistance permits to describe various factors which influence the heat transfer, as roughness, liquid or oxide layers at the contacts, or the heat transfer by radiation outside the direct contacts.

The main problem of this method is to find suitable trial functions to insert into the corresponding variational principles. The better the trial functions fit the actual temperature and flow distributions, the narrower are the bounds. If only little statistical information about the geometry of the contacts is available, upper and lower bounds are comparatively far from one another. On the contrary, for specialized geometries better bounds can be obtained by using well adapted trial functions.

In order to illustrate the method the formulism has been applied to circular contacts of equal size distributed at random. The upper bound of the effective contact resistance is obtained as a functional of the pair distribution of the circle centres. A lower bound of a similarly simple form could not been found yet. Presumably, rigorous lower bounds will always involve higher distribution functions of the contact spots. An upper bound for arbitrary geometry and distribution of the contact spots will be derived in a subsequent paper.

REFERENCES

1. K. Sanokawa, Heat Transfer between Metallic Surfaces in Contact, *Bull. J.S.M.E.* **11**, No. 44, 253–293 (1968).
2. R. Holm, *Electric Contacts*, Springer, Berlin (1967).
3. H. Y. Wong, Fundamental Studies of the Thermal Conductance of Metallic Contacts, *Proc. 8th Conf. on Thermal Conductivity*, West Lafayette, pp. 495–511 (Oct. 1968).
4. D. R. Jeng, Thermal Contact Resistance in Vacuum, *Trans. A.S.M.E.* **89C**(3), 275–276 (1967).
5. J. A. Greenwood, Constriction Resistance and the Real Area of Contact, *Br. J. appl. Phys.* **17**, 1621–1632 (1966).
6. A. M. Clausing and B. T. Chao, Thermal Contact Resistance in a Vacuum Environment, *Trans. A.S.M.E.* **87C**(2), 98–108, (1965).
7. T. Tsukizoe and T. Hisakado, On the Mechanism of Heat Transfer between Metal Surface in Contact, *Heat Transfer—Jap. Res.* **1**(1), 104–112 (1972).
8. V. M. Popov, Determination of Thermal Contact Resistance of Finished Metallic Wavy Surfaces, *J. engng Physics* **32**, 779–785 (1977) (in Russian).

9. P. H. Dederichs and R. Zeller, Variational Treatments of the Elastic Constants of Disordered Materials, *Z. Physik* **259**, 103–116 (1973).
10. E. Kröner and H. Koch, Effective Properties of Disordered Materials, *SM Archs* **1**, 183–238 (1976).
11. L. D. Landau and E. M. Lifschitz, Electrodynamics of Continuous Media, *Course of Theoretical Physics*, Vol. 8, p. 31. Akademie, Berlin (1974) (in German).

LIMITES VARIATIONELLES POUR LA RESISTANCE DE CONTACT THERMIQUE EFFECTIVE ENTRE DEUX CORPS A SURFACES RUGUEUSES

Resumé—Le présent article est consacré à l'évaluation de la résistance de contact entre surfaces rugueuses. La rugosité est décrite par une résistance de contact locale variant de manière stochastique. Des principes variationels généraux permettant de déterminer des limites supérieures et inférieures de la résistance de contact macroscopique sont déduits. Pour illustrer la méthode, on calcule une limite supérieure pour une distribution irrégulière de taches de contact circulaires à diamètres egaux. La limite de la résistance de contact macroscopique est obtenue comme une fonctionelle de la fonction de distribution binaire des taches de contact.

VARIATIONSGRENZEN FÜR DEN EFFEKTIVEN WÄRMEÜBERGANGSWIDERSTAND ZWISCHEN KÖRPERN MIT RAUHEN OBERFLÄCHEN

Zussamenfassung—Die vorliegende Arbeit befaßt sich mit der Berechnung des Übergangswiderstandes zwischen rauhen Oberflächen. Die Rauhigkeit wird durch einen stochastisch variierenden lokalen Kontaktwiderstand beschrieben. Es werden allgemeingültige Variationsprinzipien abgeleitet, die es gestatten, obere und untere Grenzen für den makroskopischen Übergangswiderstand zu bestimmen. Zur Illustration dieser Methode wird eine obere Grenze für eine beliebige Anordnung von kreisförmigen Kontakten gleicher Größe berechnet. Die für den markroskopischen Übergangswiderstand erhaltene Grenze stellt ein Funktional der Paarverteilungsfunktion der Kontakte dar.

ВАРИАЦИОННЫЕ ГРАНИЦЫ ДЛЯ ЭФФЕКТИВНОГО КОНТАКТНОГО ТЕПЛОСОПРОТИВЛЕНИЯ МЕЖДУ ТЕЛАМИ С ШЕРОХОВАТЫМИ ПОВЕРХНОСТЯМИ

Аннотация — В настоящей работе рассматривается контактное сопротивление между неровными поверхностями. Неровность учитывается как случайно меняющееся локальное сопротивление. Получены общие вариационные принципы позволяющие определить верхние и нижние границы макроскопического контактного сопротивления. В качестве примера вычислена верхняя граница для случайного распределения круглых областей соприкосновения одинаковой величины. Граница макроскопического контактного сопротивления является функционалом парной функции распределения областей соприкосновения.